

# On Estimating the Phase of a Periodic Waveform in Additive Gaussian Noise – Part II

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*Motivated by advances in signal processing technology that support more complex algorithms, a new look is taken at the problem of estimating the phase and other parameters of a periodic waveform in additive Gaussian noise. In Part I, the general problem was introduced and the maximum a posteriori probability criterion with signal space interpretation was used to obtain the structures of optimum and some suboptimum phase estimators for known constant frequency and unknown constant phase with an a priori distribution. In Part II, optimal algorithms are obtained for some cases where the frequency is a parameterized function of time with the unknown parameters and phase having a joint a priori distribution. In the last section, the intrinsic and extrinsic geometry of hypersurfaces is introduced to provide insight to the estimation problem for the small noise and large noise cases.*

## I. Introduction

The results of Part I (Ref. 1) are limited to the single-parameter estimation of phase with frequency  $f_c$  exactly known, based on an observation of duration  $T = t_2 - t_1$ . The maximum a posteriori probability (MAP) estimator  $\hat{x}_0$  is the value of the phase  $x$  that maximizes

$$\frac{2}{N_0} \int_{t_1}^{t_2} z(\tau) y(f_c \tau + x) d\tau + \ln f_x(x) \quad (17)^1$$

<sup>1</sup>Equation number from Part I.

where  $N_0$  is the one-sided power spectral density of the additive Gaussian noise,  $z(\cdot)$  is the observed noisy signal,  $y(\cdot)$  is the periodic waveform, and  $f_x(\cdot)$  is the a priori probability density of the random phase. Appropriate signal space geometry is introduced and phase estimation results obtained for general periodic waveforms including, in particular, the squarewave case.

In what follows, the estimation model is extended to include phase, which is a polynomial function of time of given degree with coefficients to be estimated. The first-degree two-dimensional case (unknown frequency) is solved for small and large noise, and concepts of differential geometry are intro-

duced to better understand and solve higher dimensional cases. The section and equation numbers continue from Part I.

## VII.2 Estimation of Phase and Frequency When the Frequency is Unknown but Constant

Using Eq. (13b)<sup>1</sup> we have, instead of Eq. (17),

$$\frac{2}{N_0} \int_{t_1}^{t_2} z(t) y(ft + x) dt + \ln f_f(f) + \ln f_x(x) \quad (79)^2$$

as the quantity to be maximized with respect to  $f$  and  $x$  to provide the MAP estimators  $\hat{f}_0$  and  $\hat{x}_0$ . Then Eq. (21) is replaced by

$$2 \langle \vec{z}, \vec{y}(f, x) \rangle + N_0 \ln f_f(f) + N_0 \ln f_x(x) \quad (80)$$

where

$$\vec{z} = \vec{y}(f_0, x_0) + \vec{n} \quad (81)$$

and

$$y(ft + x) = A_c \cos 2\pi(ft + x) \quad (82)$$

for the present. In the previous cases of known frequency,  $T$  could be chosen to make  $fT$  an integer and therefore make  $\|\vec{y}(f, x)\|$  and  $\|\vec{z}\|$  not functions of  $x$ . In the present case of unknown  $f$ , this is not possible and  $\vec{y}(f, x)$  lies on a closed hypersurface which is in effect the surface of a hypersphere with ripples. However, if we assume  $fT \gg 1$ , as is often the case of interest, then the fractional variation in  $\|\vec{y}(f, x)\|$  approaches zero and

$$\begin{aligned} \lim_{fT \rightarrow \infty} \frac{1}{T} \|\vec{y}(f, x)\| &= \lim_{fT \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} A_c^2 \cos^2 2\pi(ft + x) dt \\ &= \frac{A_c^2}{2} \end{aligned} \quad (83)$$

The vector  $\vec{y}(f, x)$  describes a two-dimensional curved surface parameterized by  $f$  and  $x$ . For fixed  $f$ , the cross section of the surface is a circle lying in a plane through the origin. The

$f, x$  coordinate system on the surface is locally orthogonal and the surface is embedded in an infinite-dimensional linear space. Some feeling for the curvature of the surface is given by Fig. 1. Calculation shows that

$$\langle \vec{y}(f, x), \vec{y}\left(f + \frac{m}{T}, x\right) \rangle = 0 \quad (84a)$$

for  $n$  any nonzero integer and

$$\langle \vec{y}(f, x), \vec{y}\left(f, x + \frac{1}{4}\right) \rangle = 0 \quad (84b)$$

Figure 1 indicates that in going from  $\vec{y}(f, x)$  to  $\vec{y}(f + 1/T, x)$ ,  $\vec{y}(f + 1/2T, x)$  leaves the plane of  $\vec{y}(f, x)$  and  $\vec{y}(f + 1/T, x)$  by an angle of  $25.8^\circ$  while making angles of  $50.5^\circ$  with  $\vec{y}(f, x)$  and  $\vec{y}(f + 1/T, x)$ . (In this signal space there are many directions normal to the above plane and the normal  $N$  dropped from  $\vec{y}(f + 1/2T, x)$  to the plane is only one of them.)

Taking the hint from Eqs. (84), a convenient orthonormal basis  $\vec{u}_i, \vec{v}_i$  for the infinite dimensional flat (euclidean) space  $L_y$  spanned by the two-dimensional curved surface described by  $\vec{y}(f, x)$  is given by

$$u_i(t) = \sqrt{\frac{2}{T}} \cos 2\pi \left(f_c + \frac{i}{T}\right) t \quad (85a)$$

$$v_i(t) = \sqrt{\frac{2}{T}} \sin 2\pi \left(f_c + \frac{i}{T}\right) t \quad (85b)$$

with  $i$  running over all integers, the time interval is  $t_1 = -T/2$ ,  $t_2 = T/2$ , and  $f_c T \gg 1$ . For the case of Eq. (82),

$$\vec{y}(f, x) = \sum_{i=-\infty}^{\infty} (y_{ui} \vec{u}_i + y_{vi} \vec{v}_i) \quad (86)$$

where

$$y_{ui}(f, x) = A_c \sqrt{\frac{T}{2}} (\cos 2\pi x) \frac{\sin \pi[(f_c - f)T + i]}{\pi[(f_c - f)T + i]} \quad (87a)$$

$$y_{vi}(f, x) = A_c \sqrt{\frac{T}{2}} (\sin 2\pi x) \frac{\sin \pi[(f_c - f)T + i]}{\pi[(f_c - f)T + i]} \quad (87b)$$

<sup>2</sup>Section and equation numbers continue from Part I.

If we define

$$I_i = \int_{-T/2}^{T/2} z(t) \cos 2\pi \left( f_c + \frac{i}{T} \right) t dt \quad (88a)$$

$$Q_i = \int_{-T/2}^{T/2} z(t) \sin 2\pi \left( f_c + \frac{i}{T} \right) t dt \quad (88b)$$

then the coordinates  $z_{ui}, z_{vi}$  of  $\vec{z}$  are given by

$$z_{ui} = \langle \vec{z}, \vec{u}_i \rangle = \sqrt{\frac{2}{T}} I_i \quad (89a)$$

$$z_{vi} = \langle \vec{z}, \vec{v}_i \rangle = \sqrt{\frac{2}{T}} Q_i \quad (89b)$$

Now

$$\langle \vec{z}, \vec{y}(f, x) \rangle = \sum_{i=-\infty}^{\infty} (z_{ui} y_{ui} + z_{vi} y_{vi}) \quad (90)$$

Substituting from Eqs. (87) and (89) gives

$$\begin{aligned} \langle \vec{z}, \vec{y}(f, x) \rangle &= \sum_{i=-\infty}^{\infty} A_c (I_i \cos 2\pi x + Q_i \sin 2\pi x) \\ &\times \frac{\sin \pi [(f_c - f)T + i]}{\pi [(f_c - f)T + i]} \end{aligned} \quad (91)$$

for substitution in Eq. (80). This is then maximized to find the MAP estimate of  $\hat{x}_0$  and  $\hat{f}_0$ .

For the case of small noise, the a priori probability terms in Eq. (80) generally become negligible and the MAP estimator  $\hat{f}_0, \hat{x}_0$  is obtained by maximizing  $\langle \vec{z}, \vec{y}(f, x) \rangle$  in Eq. (91). Following the argument on page 159 of Part I, Eq. (54) is replaced by

$$\sigma^2(\hat{x}_0) = \frac{\frac{N_0}{2}}{\left( \frac{\partial s}{\partial x} \right)^2} \quad (92)$$

$$\sigma^2(\hat{f}_0) = \frac{\frac{N_0}{2}}{\left( \frac{\partial s}{\partial f} \right)^2} \quad (93)$$

Now

$$\left( \frac{\partial s}{\partial x} \right)^2 = \left\| \frac{\partial}{\partial x} \vec{y}(f, x) \right\|^2 = \int_{-T/2}^{T/2} \left[ \frac{\partial}{\partial x} y(ft + x) \right]^2 dt \quad (94)$$

and by Eq. (82),

$$\left( \frac{\partial s}{\partial x} \right)^2 = 4\pi^2 A_c^2 \int_{-T/2}^{T/2} \sin^2 2\pi(ft + x) dt = 2\pi^2 A_c^2 T \quad (95)$$

since  $T \gg 1/f$ . Then by Eq. (92),

$$\sigma^2(\hat{x}_0) = \frac{\frac{N_0}{2}}{2\pi^2 A_c^2 T} = \frac{1}{4\pi^2} \frac{N_0}{A_c^2 T} \quad (96)$$

which is the same as the result in Eq. (54) for a sinusoid of known frequency. Thus for the small noise case, the phase error is not increased if the frequency is unknown (but constant).

Proceeding with the estimate  $\hat{f}_0$  of the frequency  $f_0$  we have

$$\left( \frac{\partial s}{\partial f} \right)^2 = \left\| \frac{\partial}{\partial f} \vec{y}(f, x) \right\|^2 = \int_{-T/2}^{T/2} \left[ \frac{\partial}{\partial f} y(ft + x) \right]^2 dt \quad (97)$$

and by Eq. (82),

$$\left( \frac{\partial s}{\partial f} \right)^2 = 4\pi^2 A_c^2 \int_{-T/2}^{T/2} t^2 \sin^2 2\pi(ft + x) dt = \frac{\pi^2}{6} A_c^2 T^3 \quad (98)$$

since  $T \gg 1/f$ . Then by Eq. (93),

$$\sigma^2(\hat{f}_0) = \frac{3}{\pi^2} \frac{N_0}{A_c^2 T^3} \quad (99)$$

From Eqs. (96) and (99),

$$T = 2\sqrt{3} \frac{\sigma(\hat{x}_0)}{\sigma(\hat{f}_0)} \quad (100)$$

$$\frac{P}{N_0} = \frac{A_c^2/2}{N_0} = \frac{1}{16\sqrt{3}\pi^2} \cdot \frac{\sigma(\hat{f}_0)}{\sigma^3(\hat{x}_0)} \quad (101)$$

For example, with  $\sigma(\hat{x}_0) = 0.01$  cycles ( $3.6^\circ$ ) and  $\sigma(\hat{f}_0) = 1$  Hz we have  $T = 0.0346$  s and  $P/N_0 = 3656$  (35.6 dB). The frequency spacing  $1/T$  of the correlators in Eq. (88) would then be 28.87 Hz. Both the frequency spacing and the required  $P/N_0$  are directly proportional to  $\sigma(\hat{f}_0)$ .

Returning now to the general case, let us consider the maximization of Eq. (80). For the initial estimate, we generally have  $f_x(\cdot) = 1$  and  $f_f(\cdot)$  constant over some interval and zero elsewhere. Thus let us consider the maximization of Eq. (91). Define

$$\alpha = (f_c - f)T \quad (102)$$

and

$$I(\alpha) = \sum_{i=-\infty}^{\infty} I_i \frac{\sin \pi(\alpha - i)}{\pi(\alpha - i)} \quad (103a)$$

$$Q(\alpha) = \sum_{i=-\infty}^{\infty} Q_i \frac{\sin \pi(\alpha + i)}{\pi(\alpha + i)} \quad (103b)$$

so that Eq. (91) becomes

$$\langle \vec{z}, \vec{y}(f, x) \rangle = A_c [I(\alpha) \cos 2\pi x + Q(\alpha) \sin 2\pi x] \quad (104)$$

$I(\alpha)$  and  $Q(\alpha)$  are simply the cardinal series (minimum bandwidth) interpolations of the samples  $I_i$  and  $Q_i$  respectively, with a sampling rate of unity in the  $\alpha$  coordinate. Now by Eqs. (81), (82), (86), (89), and (68) the  $I_i$  and  $Q_i$  are simply  $\sqrt{T/2} y_{0ui}$  and  $\sqrt{T/2} y_{0vi}$ , respectively, plus independent Gaussian random variables of mean zero and variance  $N_0 T/4$ . By Eq. (87)

$$\sqrt{T/2} y_{0ui} = \frac{A_c T}{2} (\cos 2\pi x_0) \frac{\sin \pi(\alpha_0 + i)}{\pi(\alpha_0 + i)} \quad (105a)$$

$$\sqrt{T/2} y_{0vi} = \frac{A_c T}{2} (\sin 2\pi x_0) \frac{\sin \pi(\alpha_0 + i)}{\pi(\alpha_0 + i)} \quad (105b)$$

Thus,  $I_i$  and  $Q_i$  are samples of

$$\frac{A_c T}{2} (\cos 2\pi x_0) \frac{\sin \pi(\alpha - \alpha_0)}{\pi(\alpha - \alpha_0)} \quad (106a)$$

and

$$\frac{A_c T}{2} (\sin 2\pi x_0) \frac{\sin \pi(\alpha - \alpha_0)}{\pi(\alpha - \alpha_0)} \quad (106b)$$

respectively, taken at integer values of  $\alpha$  plus the random variables described above. Now the function of  $\alpha$  in Eq. (106) is band limited at half the sampling rate and so the interpolated functions of Eq. (103) are simply the functions of Eq. (106) plus Gaussian band-limited white noise of bandwidth  $1/2$  and variance  $N_0 T/4$ . Thus

$$I(\alpha) = \frac{T}{2} \left[ A_c (\cos 2\pi x_0) \frac{\sin \pi(\alpha - \alpha_0)}{\pi(\alpha - \alpha_0)} + n_I(\alpha) \right] \quad (107a)$$

$$Q(\alpha) = \frac{T}{2} \left[ A_c (\sin 2\pi x_0) \frac{\sin \pi(\alpha - \alpha_0)}{\pi(\alpha - \alpha_0)} + n_Q(\alpha) \right] \quad (107b)$$

where  $n_I(\alpha)$  and  $n_Q(\alpha)$  are independent Gaussian band-limited white random processes with

$$\sigma^2 [n_I(\alpha)] = \sigma^2 [n_Q(\alpha)] = \frac{N_0}{2} \quad (108)$$

Returning now to Eq. (104) we observe that  $\langle \vec{z}, \vec{y}(f, x) \rangle$  can be maximized in two steps. In the first step we choose  $x(\alpha)$  to maximize the expression for each  $\alpha$ . A necessary condition is

$$\frac{\partial}{\partial x} \langle \vec{z}, \vec{y}(f, x) \rangle = 2\pi A_c [-I(\alpha) \sin 2\pi x + Q(\alpha) \cos 2\pi x] = 0 \quad (109)$$

which gives

$$\tan 2\pi x = \frac{Q(\alpha)}{I(\alpha)} \quad (110)$$

This has two solutions for  $x$ , one giving the absolute maximum and the other the absolute minimum of Eq. (104) with respect to  $x$ . The value of  $x$  in Eq. (110) giving the maximum is that which causes  $\cos 2\pi x$  to have the same sign as  $I(\alpha)$  and  $\sin 2\pi x$  to have the same sign as  $Q(\alpha)$ . From Eq. (110) and the above,

$$\sin 2\pi \hat{x}_0 = \frac{Q(\alpha)/|I(\alpha)|}{\sqrt{1 + \left(\frac{Q(\alpha)}{I(\alpha)}\right)^2}} \quad (111a)$$

$$\cos 2\pi \hat{x}_0 = \frac{\text{sgn}[I(\alpha)]}{\sqrt{1 + \left(\frac{Q(\alpha)}{I(\alpha)}\right)^2}} \quad (111b)$$

Substitution in Eq. (104) gives

$$\langle \vec{z}, \vec{y}(f, \hat{x}_0) \rangle = A_c \sqrt{I^2(\alpha) + Q^2(\alpha)} \quad (112)$$

Thus, by Eq. (102)  $\hat{f}_0 = f_c - (\hat{\alpha}_0/T)$  where  $\hat{\alpha}_0$  is the value of  $\alpha$  which maximizes

$$\frac{(\vec{z}, \vec{y}(f, \hat{x}_0))^2}{A_c^2} = I^2(\alpha) + Q^2(\alpha) \quad (113)$$

The large noise behavior of  $\hat{\alpha}_0$  may be studied by using Eq. (107) to substitute for  $I(\alpha)$  and  $Q(\alpha)$  in Eq. (113). The actual estimator must use Eqs. (103), (113), and (111). Even in the absence of noise, Eq. (113) has many stationary points (relative maxima and minima) and so an indirect estimator is a problem. Since the maxima and minima of Eq. (113) are always stationary points ( $n_I(\alpha)$  and  $n_Q(\alpha)$  have all derivatives), one approach would be to solve

$$\frac{d}{d\alpha} (I^2(\alpha) + Q^2(\alpha)) = 0 \quad (114)$$

for all of the stationary points in the a priori interval determined by  $f_f(\alpha)$  and then evaluate Eq. (113) at these points to locate the absolute maximum. For a particular noise realization, Eq. (112) will generally not be an even function about its maximum and so the MAP estimator is generally not the minimum mean square error (MSE) or mean absolute error (MAE) estimator. In the case of a general a priori distribution  $f_f(\cdot)$  on  $f$  and noise which is not small, it is necessary to substitute Eq. (112) in Eq. (80) and seek the maximum of

$$2A_c \sqrt{I^2(\alpha) + Q^2(\alpha)} + N_0 \ln f_\alpha(\alpha) \quad (115)$$

where  $f_\alpha(\cdot)$  is easily obtained from  $f_f(\cdot)$  in the usual way.

## VIII. Some Geometrical Aspects

In the more general case where Eq. (82) is replaced by

$$y \left( x + ft + \frac{f^2 t^2}{2} + \dots + f^{(N-2)} \frac{t^{N-1}}{(N-1)!} \right) \quad (116)$$

then the terminal points of

$$\vec{y}(x, f, \dot{f}, \dots, f^{(N-2)}) \quad (117)$$

form an  $N$ -dimensional subspace (generally nonlinear) that is coordinatized by the  $N$  parameters to be estimated and that is imbedded in the infinite dimensional linear space<sup>3</sup> in which the observed signal plus noise is represented. As we have seen, the estimation problem involves such questions as the point in

the parameter subspace that is at minimum distance from an arbitrary point in the imbedding linear signal (plus noise) space and the effect of the noise on the location of the point in the parameter subspace. The solution of the estimation problem depends on both the intrinsic (Ref. 3) and extrinsic geometrical properties of the nonlinear parameter space with the small noise results depending only on the intrinsic geometry. To gain insight, we examine several cases.

*First case:* Phase estimation only with known frequency  $f_c$  ( $N = 1$ ). This was treated analytically in Section VI (Part I) with the periodic waveform given by Eq. (55). The geometrical situation is illustrated in Fig. 2. The one-dimensional parameter space (path) lies on the surface of a  $2n$  dimensional hypersphere (and is therefore nonlinear). For this one-dimensional case, the intrinsic geometry of the path is completely described by the single scalar  $ds/dx$ , given by Eq. (56), with the small noise result given by the second term of Eq. (54). If  $y(\cdot)$  has one or more discontinuities, such as a square wave, then  $ds/dx \rightarrow \infty$  and the small-noise performance becomes arbitrarily good (the closed path, while continuous, has no derivative and is infinite in length).

For large noise, the extrinsic geometry of the path becomes important. The radius of curvature (ROC) (Ref. 4) is given by (see Eq. 55, Part I)

$$ROC = \sqrt{\frac{T}{2}} \frac{\sum_{j=1}^n j^2 (a_j^2 + b_j^2)}{\sqrt{\sum_{j=1}^n j^4 (a_j^2 + b_j^2)}} \quad (118)$$

which vanishes if there is a discontinuity in  $y(\cdot)$  or its derivatives. The distance between any two points on the path,  $\vec{y}(x_1)$  and  $\vec{y}(x_2)$ , is given by

$$\|\vec{y}(x_1) - \vec{y}(x_2)\|^2 = T \sum_{j=1}^n (a_j^2 + b_j^2) [1 - \cos 2\pi j(x_1 - x_2)] \quad (119)$$

which, of course, never exceeds twice the radius of the hypersphere in whose surface the path is imbedded. Note that the above extrinsic properties are invariant over the path.

*Second case:* Phase and frequency estimation with unknown constant frequency ( $N = 2$ ) and sinusoidal waveform. This was treated analytically in Section VII. The two-dimensional parameter space (surface) is nonlinear as indicated by Fig. 1. Even for the special case of  $y(\cdot)$  sinusoidal, the two-

<sup>3</sup>An  $N$ -dimensional noneuclidean space can always be imbedded in euclidean space of not more than  $N(N+1)/2$  dimensions (Ref. 2).

dimensional surface spans (must be imbedded in) an infinite dimensional signal space. It is something like a tube extending through all of the dimensions of the signal space as  $f$  runs over its range. The intrinsic geometry of the surface is completely described by (Ref. 5)

$$ds^2 = \sum_{i,j=1}^2 g_{ij} du^i du^j \quad (120)$$

where  $s$  is the distance in the surface,  $u^1 = x$  and  $u^2 = f$  are the contravariant coordinate values in the surface, and  $g_{ij}$  is the second-order covariant metric tensor of the surface. Calculation gives

$$\begin{aligned} g_{11} &= 2\pi^2 A_c^2 T \\ g_{22} &= \frac{\pi^2 A_c^2 T^3}{6} \\ g_{12} &= g_{21} = 0 \end{aligned} \quad (121)$$

Thus the  $f, x$  coordinate system in the surface is everywhere orthogonal ( $g_{12} = g_{21} = 0$ ) and uniform ( $g_{11}$  and  $g_{22}$  are constant). Also the surface is developable, that is, it has no intrinsic curvature. Now

$$\left(\frac{ds}{dx}\right)^2 = g_{11}, \quad \left(\frac{ds}{df}\right)^2 = g_{22} \quad (122)$$

and the small noise result has been given in Eqs. (96) and (99).

The extrinsic geometry of the surface is given by Eqs. (86) and (87) and the large noise estimation case has been considered beginning with Eq. (102).

*Third case:* Phase, frequency, and frequency-rate estimation with unknown constant frequency rate ( $N = 3$ ). The surface of the second case now becomes a three-dimensional volume by the coordinate  $\dot{f}$ , which turns out not to be orthogonal to the surface coordinatized by  $x$  and  $f$ . For small noise, the intrinsic geometry is determined by

$$ds^2 = \sum_{i,j=1}^3 g_{ij} du^i du^j \quad (123)$$

with  $u^1 = x$ ,  $u^2 = f$ ,  $u^3 = \dot{f}$ .  $g_{11}$ ,  $g_{22}$ ,  $g_{12}$ , and  $g_{21}$  have the same values as in the second case. Calculation gives

$$g_{33} = \frac{\pi^2 A_c^2 T^5}{160}$$

$$g_{23} = g_{32} = 0$$

$$g_{13} = g_{31} = \frac{\pi^2 A_c^2 T^3}{12} \quad (124)$$

Thus the  $\dot{f}$  coordinate is orthogonal to the  $f$  coordinate and the angle  $\theta$  between the  $\dot{f}$  and  $x$  coordinates is given by

$$\begin{aligned} \cos \theta &= \frac{g_{13}}{\sqrt{g_{11} g_{33}}} = \frac{\sqrt{5}}{3} = 0.745 \\ \theta &= 42^\circ \quad \sin \theta = \frac{2}{3} \end{aligned} \quad (125)$$

So in the small noise case, the errors in the estimates  $\hat{f}_0$  and  $\hat{x}_0$  have a correlation of 0.745 and the errors in the other two pairs of estimates are uncorrelated. As in the two-dimensional case, the intrinsic geometry of the three-dimensional manifold has no curvature ( $g_{ij}$  components are constants).

For the small noise case

$$\sigma^2(\hat{f}_0) = \frac{N_0/2}{\left(\frac{ds}{df}\right)^2} = \frac{N_0/2}{g_{33}} \quad (126)$$

giving

$$\sigma^2(\hat{f}_0) = \frac{80}{\pi^2} \frac{N_0}{A_c^2 T^5} \quad (127)$$

For the example following Eq. (101), where it was found that  $\sigma(\hat{x}_0) = 0.01$  cycles and  $\sigma(\hat{f}_0) = 1$  Hz, with  $T = 0.0346$  s and  $P/N_0 = 3656$ , the result for  $\hat{f}_0$  is

$$\sigma(\hat{f}_0) = 149.5 \text{ Hz/s}$$

If  $T$  is increased by a factor of 10 to 0.346 s, the above becomes

$$\sigma(\hat{f}_0) = 0.47 \text{ Hz/s}$$

The above results can be extended in a straightforward way to the estimation of the higher derivatives of frequency ( $N > 3$ ) if a priori knowledge indicates they are significant.

## IX. Algorithms

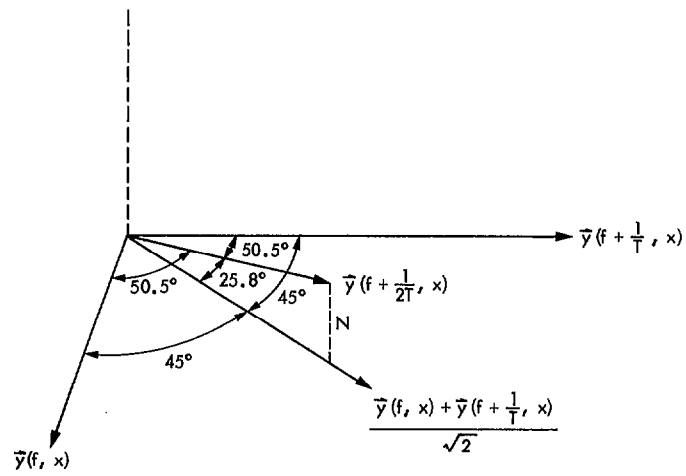
Computational algorithms based on the preceding results such as Eqs. (88), (103), (113), (111), and (102) are feasible with current digital signal processing technology and would

provide a sequence of MAP estimates at a rate  $1/T$ . Such algorithms are quite different from phase-locked loops, which are frequently used for phase and frequency estimation. The phase-locked loop is an invented causal dynamical system that does not provide an optimum estimate based on a criterion.

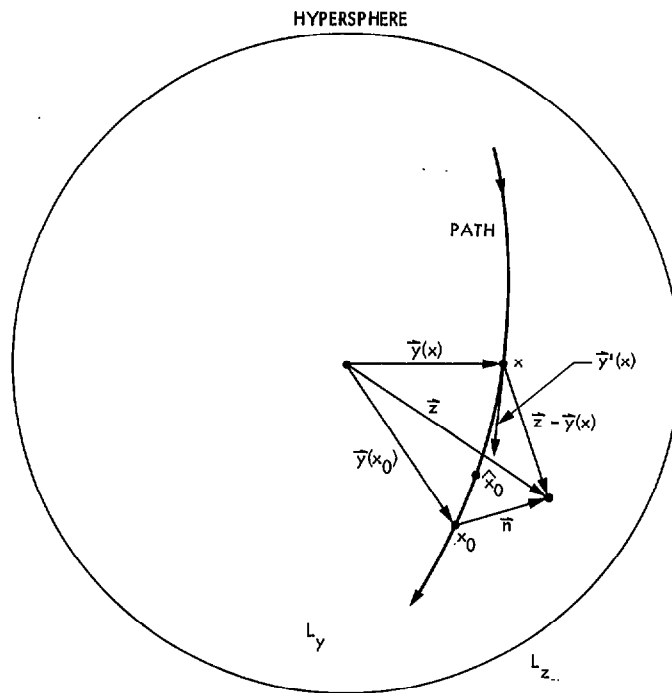
The phase-locked loop, like any real feedback system, has its own dynamics and stability considerations, which are not really a part of the estimation problem. The stability considerations of third- and higher-order phase-locked loops are not present in algorithms based on the results of this paper.

## References

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4. Ibid., Section 4.
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**Fig. 1. Surface curvature**



$\hat{x}_0$  IS POINT ON PATH NEAREST TO  $\vec{z}$   
 $\hat{x}_0$  MINIMIZES ANGLE BETWEEN  $\vec{y}(\hat{x}_0)$  AND  $\vec{z}$

**Fig. 2. Paths on hypersphere**